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Note

The kissing number of the regular polygon

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Abstract

Let P_n be an arbitrary regular polygon with n sides. What is the maximum number $k(P_n)$ of congruent regular polygons (copies of P_n) that can be arranged so that each touches P_n but no two of them overlap? Youngs (1939), Klamkin (1995) and others established that $k(P_3) = 12$, $k(P_4) = 8$ and $k(P_6) = 6$. In this paper, we will establish the general and nice result $k(P_n) = 6$, where $n > 6$. ©1998 Elsevier Science B.V. All rights reserved

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1. Introduction

Two plane figures are said to kiss each other if they do not overlap but their boundaries have nonempty intersection. For a plane figure F, a kissing configuration of order n consists of $n+1$ congruent nonoverlapping copies of F such that one copy kisses each of the remaining n. The kissing number of F,denoted by $k(F)$, is the largest integer n such that a kissing configuration of order n exists for F.

Let P_n be an arbitrary regular polygon with n sides. For the square P_4, Youngs [1] seems to have been the first to establish that $k(P_4) = 8$ and Fig. 1 is a kissing configuration of order $k(P_4) = 8$ for the square P_4.

The determination of $k(P_4)$ was later posed as a Putnam competition problem. Youngs′ result was reproduced in the solution book [3]. Klamkin et al. [2] also gave an elementary proof that $k(P_4) = 8$ and presented a few kissing configurations of order $k(P_4) = 6$ for P_6 (of which Fig. 2 is only one).

The kissing number of every plane figure is at least 6. In fact, there is always a kissing configuration in which all 7 copies are translates of one another (see [4]). The circle shows that the universal lower bound of 6 cannot be increased. On the other hand, there is no universal upper bound, as a kissing configuration of order $2n+6$

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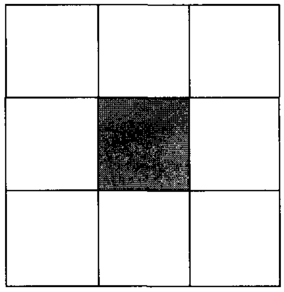

Fig. 1.

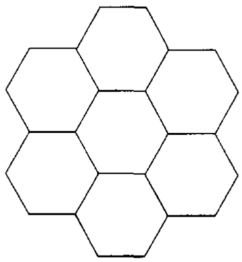

Fig. 2.

exists for the 1 by n rectangle, where n is an arbitrary positive integer. For the regular polygons P_n, we only know that $k(P_3) = 12$, $k(P_4) = 8$ and $k(P_6) = 6$ so far. Given an arbitrary regular polygon P_n, what does $k(P_n)$ equal? In this paper, we treat this question. The proofs in [1,2] are complicated. The proof in this paper is quite simple, and it is based on exploring the maximum and minimum of the visual angle, where the angle is the angle of the centers of two adjacent regular polygons to the center of the central one.

2. The theorem and its proof

Theorem. *Suppose that P_n is an arbitrary regular polygon with n sides and $n > 6$. Then*

$$k(P_n) = 6.$$

In order to prove the above theorem, we first prove the following Lemma 1.

Lemma 1. *Let O be the center of the regular polygon $P_n (n \geqslant 4)$. Let A and B be the centers of any two nonoverlapping regular polygons P'_n, P''_n that kiss P_n such that P'_n, P''_n kiss each other and P'_n, P''_n, P_n are congruent. If angle $AOB = \theta_n$, then*

$$\theta_n \geqslant \arccos\left[\left(3 - \cos\frac{2\pi}{n}\right)/4\right].$$

Proof. Without loss of generality, we suppose that $R = 1$; then $r_n = \cos(\pi/n)$, where R is the radius of P_n, P_n' and P_n'', r_n is the distance from the centers of P_n, P_n', P_n'' to their sides. Clearly, both OA and OB are at least $2r_n = 2\cos(\pi/n)$ and at most $2R = 2$, and AB is at least $2r_n$.

Since $AB \geqslant 2r_n = 2\cos(\pi/n)$, if we replace the triangle AOB by the triangle A_1OB_1 with $OA_1 = OA, OB_1 = OB, A_1B_1 = 2\cos(\pi/n)$, then angle $AOB \geqslant$ angle A_1OB_1. Let $OA_1 = x$ and $OB_1 = y$. By the Law of Cosines, we have

$$\cos A_1OB_1 = \left(x^2 + y^2 - 4\cos^2\frac{\pi}{n}\right)/2xy = F(x,y),$$

where $2\cos(\pi/n) \leqslant x, y \leqslant 2$. Let $F(x,y) = [x^2 + y^2 - (2\cos(\pi/n))^2]/2xy$, then

$$F_x'(x,y) = \left[x^2 - y^2 + \left(2\cos\frac{\pi}{n}\right)^2\right]/2x^2y,$$

$$F_y'(x,y) = \left[y^2 - x^2 + \left(2\cos\frac{\pi}{n}\right)^2\right]/2xy^2,$$

Since $n > 4$ and $2 \geqslant x, y \geqslant 2\cos(\pi/n) > 2\sin(\pi/n)$, we have

$$x^2 - y^2 + \left(2\cos\frac{\pi}{n}\right)^2 > \left(2\sin\frac{\pi}{n}\right)^2 - 4 + \left(2\cos\frac{\pi}{n}\right)^2 = 0.$$

This inequality implies that $F_x'(x,y) > 0$ and $F_y'(x,y) > 0$. Therefore,

$$F(x,y) \leqslant F(2,y) \leqslant F(2,2) = \left(3 - \cos\frac{2\pi}{n}\right)/4.$$

Now, the above relation implies that

$$\begin{aligned}\theta_n &= \text{angle}\, AOB \geqslant \text{angle}\, A_1OB_1 \\ &= \arccos F(x,y) \geqslant \arccos F(2,2) \\ &= \arccos\left[\left(3 - \cos\frac{2\pi}{n}\right)/4\right].\end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2. *If P_n is a regular polygon then $k(P_n) \geqslant 6$* (see [4]).

Proof of the theorem. Let $\varphi_n = \arccos[(3 - \cos(2\pi/n))/4]$; when $n = 7$, we have $\cos(2\pi/7) > 0.6225, \varphi_7 = \arccos[(3 - \cos(2\pi/7))/4] > 53^\circ$, since $\theta_7 \geqslant \varphi_7$ by Lemma 1,

$$k(P_7) \leqslant 360^\circ/\varphi_7 \leqslant 360^\circ/53^\circ < 7.$$

By Lemma 2, we have $k(P_7) \geqslant 6$, thus $k(P_7) = 6$.

When $n > 7$, $\cos(2\pi/n) < \cos(2\pi/(n+1))$ implies

$$\arccos\left[\left(3 - \cos\frac{2\pi}{n}\right)/4\right] < \arccos\left[\left(3 - \cos\frac{2\pi}{n+1}\right)/4\right],$$

i.e. $\varphi_n < \varphi_{n+1}$. Hence when $n > 7$ we have $\theta_n \geqslant \varphi_n > \varphi_{n-1} > \cdots > \varphi_8 > \varphi_7$, therefore

$$k(P_n) \leqslant 360^\circ/\varphi_n < 360^\circ/\varphi_7 < 7.$$

This combined again with Lemma 2, yelds $k(P_n) = 6$.

It remains as an open problem to find $k(P_5)$, where P_5 is the regular pentagon. We know that $k(P_5) = 6$ or 7 by Lemmas 1 and 2, and easily find a kissing configuration of order 6 for the regular pentagon P_5. We guess that $k(P_5) = 6$.

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